Analysis 2, Summer 2024

List 2

Path integrals, calculations with gradients

62. Draw the curve parameterized by

$$x = 5\cos(t), \qquad y = 5\sin(t)$$

with $\frac{\pi}{2} \le t \le \pi$.

63. Draw the curve described by by $x = t,\ y = t^3 - t$ with $-2 \le t \le 2$.

64. Draw the curve described by

$$\vec{r} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} t$$

with $0 \le t \le 1$.

A **parameterization** of a curve is a continuous vector function $\vec{r} : [a, b] \to \mathbb{R}^n$, where $[a, b]$ is some interval.

65. Give a parameterization of the line segment from that starts at $(0, 0)$ and ends at $(7, 2)$.

66. Calculate $\displaystyle\int_C f \, \mathrm{d}s$ where $f(x, y) = x - y$ and C is the line segment that starts at $(2, 0)$ and ends at $(4, 5)$.

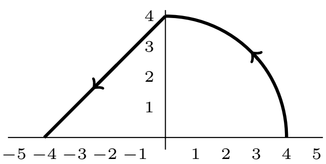
67. Integrate $x - y$ along the line segment from $(2, 0)$ to $(4, 5)$.

68. Integrate xe^y along the half-circle $\{(x, y) : x^2 + y^2 = 1,\ x \ge 0\}$.

69. (a) Integrate $f(x, y) = \sin(\pi y)$ along the line segment from $(0, 1)$ to $(1, 0)$.
    (b) Integrate $f(x, y) = \sin(\pi y)$ along the line segment from $(1, 0)$ to $(0, 1)$.
    (c) Compare your answers to parts (a) and (b).

70. Calculate $\displaystyle\int_C (2yx^2 - 4x) \, \mathrm{d}s$ where C is the bottom half of the circle of radius 3 centered at the origin.

71. Integrate $f(x, y) = y$ along the path shown here:



72. (a) Integrate $f(x, y, z) = xy + z$ along the "helix" curve $\vec{r} = [\cos(t), \sin(t), t]$ with $0 \le t \le 4\pi$.
    (b) Integrate $f(x, y, z) = xy + z$ along the line segment from $(1, 2, 3)$ to $(4, 5, 6)$.

The **gradient** of $f(x, y)$ is the vector $\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$. We write ∇f for this vector.

73. Find the gradient of e^{x+y^2} at the point $(x, y) = (-5, 2)$.

74. Calculate both $f(1, 6)$ and $\nabla f(1, 6)$ for $f(x, y) = x^4 + y\ln(x)$.

75. Give the gradient of $\cos(x+y^2)$. (This will be a 2D vector whose entries are formulas with x and y.)

76. Compute the *length* of the gradient of $x^2\sin(y)$ at the point $(4, \frac{\pi}{3})$. (This is just a number.)

77. Give ∇g for $g(x,y) = y^3\cos(xy) + \sqrt{x}$.

78. Calculate the gradient of $f(x,y,z) = xz + e^{y+z}$, which is defined as the 3D vector

$$\nabla f = \begin{bmatrix} f'_x \\ f'_y \\ f'_z \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}.$$

79. For $f = \dfrac{x}{yz}$, calculate $\left|\nabla f(1,-1,2)\right|$.

80. For $f(x,y) = \ln(x) + e^y$, calculate $(\frac{12}{13}\hat{\imath} + \frac{5}{13}\hat{\jmath}) \cdot \nabla f(4,0)$.

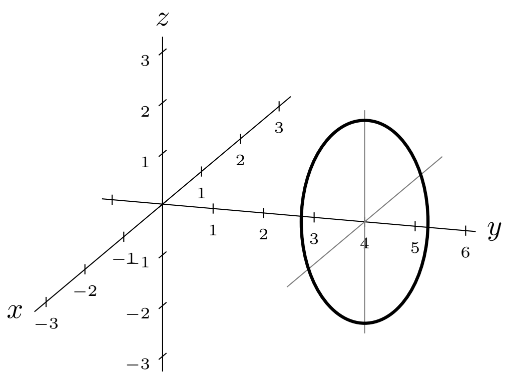
81. For $f(x,y) = \dfrac{x}{y}$, give an example of a vector that is perpendicular to $\nabla f(12,2)$.

Starred tasks (☆) use ideas or methods that are not *required for this course. But they can be interesting to think about.*

☆82. Find a function $f(x,y,z)$ for which $\nabla f = \begin{bmatrix} 2xz^3 - y\sin x \\ \cos x \\ 3x^2z^2 \end{bmatrix}$.

☆83. If $\vec{F} = \begin{bmatrix} x^3y \\ e^{yz} \\ y \end{bmatrix}$, calculate $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ using the idea that $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$.

☆84. Circles can be in 3D space! Integrate $f(x,y,z) = x^2y^2$ over the circle *in the vertical plane* $y = 4$ with center $(0,4,0)$ and radius 2.



☆85. The sets

$$A = \{(x,y) : x = \sin t,\ y = (\sin t)^2,\ 0 \le t \le \pi\}$$
$$B = \{(x,y) : x = \ln t,\ y = (\ln t)^2,\ 1 \le t \le e\}$$

are exactly the same (they are both $\{(x,y) : y = x^2,\ 0 \le x \le 1\}$). Why are

$$\int_0^\pi \sqrt{(x')^2 + (y')^2}\,\mathrm{d}t = \int_0^\pi \sqrt{(\cos t)^2 + (2\sin t\cos t)^2}\,\mathrm{d}t \approx 2.9579$$
$$\int_1^e \sqrt{(x')^2 + (y')^2}\,\mathrm{d}t = \int_1^e \sqrt{\left(\frac{1}{t}\right)^2 + \left(\frac{2\ln t}{t}\right)^2}\,\mathrm{d}t \approx 1.4789$$

not equal?